Solution to the cosmological constant problem

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Abstract

The current, accelerated, phase of expansion of our universe can be modelled in terms of a cosmological constant Λ. A key issue in theoretical physics is to explain the extremely small value of the dimensionless parameter $\Lambda L_P^2 \approx 3.4 \times 10^{-122}$, where $L_P \equiv (G\hbar/c^3)^{1/2}$ is the Planck length. We show that this value can be understood in terms of a new dimensionless parameter N, which counts the number of modes inside a Hubble volume crossing the Hubble radius, from the end of inflation until the beginning of the accelerating phase. Theoretical considerations suggest that $N \approx 4\pi$. On the other hand, N is related to $\ln(\Lambda L_P^2)$ and two other parameters which will be determined by high energy particle physics: (a) the ratio (n_γ/n_m) between the number densities of photons and matter and (b) the energy scale of inflation ($E_{\rm inf}$). For realistic values of $(n_\gamma/n_m) \approx 4.3 \times 10^{10}$ and $E_{\rm inf} \approx 10^{15}$ GeV, our postulate $N = 4\pi$ leads to the observed value of the cosmological constant. This provides a unified picture of cosmic evolution relating the early inflationary phase to the late accelerating phase.

1 Introduction and Summary

Observations indicate that our universe can be characterized by three distinct phases of evolution: An early inflationary phase, driven possibly by a scalar field, a late-time accelerated phase, dominated by dark energy, and a transient phase in between, dominated by radiation and matter (including both dark matter and baryons).

The first and the last phases have approximately constant Hubble radii, $H_{\rm inf}^{-1}$ and H_Λ^{-1}, related to the respective constant rates of expansion of the universe during these phases. While the Hubble radius $H_{\rm inf}^{-1}$ of the inflationary phase is related to the mechanism driving inflation, the Hubble radius during the late-time accelerating phase can be related to a cosmological constant Λ with $\Lambda \equiv 3H_\Lambda^2$. Using the Planck length, $L_P \equiv (G\hbar/c^3)^{1/2}$, we can now construct two dimensionless ratios $\beta^{-1} \equiv H_{\rm inf} L_P$ and ΛL_P^2. If inflation took place at

GUTs scale ($\sim 10^{15}$ GeV), then $\beta \approx 3.8 \times 10^7$, while observations [1,2] suggest that $\Lambda L_P^2 \approx 3.4 \times 10^{-122} \approx 3 \times e^{-281}$. It is generally believed that physics at, say, the GUTs scale will eventually enable us to estimate the value of β and thus understand the inflationary phase. But no fundamental principle has been suggested in the literature to explain the extremely small value of ΛL_P^2, which is related directly or indirectly to the cosmological constant problem. Understanding this issue [3] from first principles is considered very important in theoretical physics today.

Here we describe an approach which is powerful enough to tackle this problem. It is obvious that $\ln(\Lambda L_P^2)$ is a more tractable quantity than ΛL_P^2 itself. If $\ln(\Lambda L_P^2)$ can be related to a physically meaningful parameter, the value of which can be independently understood, then we obtain a handle on the numerical value of the cosmological constant. We show that $\ln(\Lambda L_P^2)$ is related to a dimensionless number (which we call the 'Cosmic Mode Index', or CosMIn) that counts the number of modes within the Hubble volume that cross the Hubble radius in the radiation and matter dominated eras — that is, between the end of inflation and the beginning of late-time acceleration. CosMIn *is a characteristic number for our universe* and it is possible to argue [4] that the natural value for N is of the order of 4π; i.e., $N = 4\pi\mu$ with μ being of order unity.

This postulate allows us to determine the numerical value of ΛL_P^2. For example, in a purely radiation dominated universe with Planck scale inflation, one finds that $\Lambda L_P^2 = (3/4)\exp(-24\pi^2\mu)$ (see Ref. [4] and Eq. (2) below) which reproduces the observed value of ΛL_P^2 when $\mu \approx 1.18$. *There are no adjustable parameters in this context.* In the real universe, this result gets modified to $\Lambda L_P^2 = C\beta^{-2}\exp(-24\pi^2\mu)$, where C depends on n_γ/n_m, the ratio between the number densities of photons and matter. *We again obtain the correct observed value of the cosmological constant* for a GUTs scale inflation and the allowed range of C permitted by cosmological observations. This is the key result of this paper.

2 CosMIn and the cosmological constant

The Hubble radius remains constant during the inflationary and the late-time accelerating phases, while it evolves as $H^{-1}(a) \propto a^2$ in the radiation dominated phase and as $H^{-1}(a) \propto a^{3/2}$ in the matter dominated phase. An important concept in standard cosmological models is the crossing of the Hubble radius by modes characterized by a co-moving wave number, k, related to the proper wavelength $\lambda_{\rm prop}(a) \equiv a/k$. This crossing occurs whenever the equation $\lambda_{\rm prop}(a) = H^{-1}(a)$, i.e., $k = aH(a)$ is satisfied.

For a *generic* mode (see Fig.1; line marked ABC), this equation has three solutions: $a = a_A$ (during the inflationary phase; point A in Fig.1), $a = a_B$ (during the radiation/matter dominated phase; point B), $a = a_C$ (during the late-time accelerating phase; point C). But modes with $k < k_-$ exit during the inflationary phase and never re-enter. Similarly, modes with $k > k_+$ remain inside the Hubble radius until very late and only exit during the late-time

acceleration phase.

The modes with comoving wavenumbers in the range $(k, k + dk)$ where $k = aH(a)$ and $dk = [d(aH)/da]da$ cross the Hubble radius during the interval $(a, a + da)$. The number of modes in a comoving Hubble volume $V_{\rm com} = (4\pi H^{-3}/3a^3)$ with wave numbers in the interval $(k, k + dk)$ is $dN = V_{\rm com} d^3k/(2\pi)^3$. Hence, the number of modes that cross the Hubble radius in the interval $(a_1 < a < a_2)$ is given by

$$\begin{aligned} N(a_1, a_2) &= \int_{a_1}^{a_2} \frac{V_{\rm com} k^2}{2\pi^2} \frac{dk}{da}\, da = \frac{2}{3\pi} \int_{a_1}^{a_2} \frac{d(Ha)}{Ha} \\ &= \frac{2}{3\pi} \ln\left(\frac{H_2 a_2}{H_1 a_1}\right), \end{aligned} \tag{1}$$

where we have used $V_{\rm com} = 4\pi/3H^3a^3$ and $k = Ha$. In the above definition we have made natural choices for some numerical factors, which, as we shall see, lead to interesting and acceptable results.

The number N has several useful properties. It is well defined for any range $(a_1 < a < a_2)$ and is invariant under multiplication of a by an arbitrary factor. It is positive when $H_2a_2 > H_1a_1$ and negative otherwise. Further, if a_1 and a_2 are chosen to be epochs at which a given mode with wave number k crosses the Hubble radius, so that $H_1a_1 = H_2a_2 = k$, then $N(a_1, a_2) = 0$. For the generic mode in Fig.1 which crosses the Hubble radius thrice, at A, B and C, the value of $N(a_A, a)$ increases from $a = a_A$ till the end of inflation at $a = a_X$, and then decreases from $a = a_X$ to $a = a_B$, reaching zero again at $a = a_B$. All the modes which exit the Hubble radius during $a_A < a < a_X$ enter the Hubble radius during $a_X < a < a_B$.

It follows that the number of modes $N(a_X, a_Y)$ which enter the Hubble radius during the radiation/matter dominated era is a *characteristic number for our universe*, which we call CosMIn and denote its *magnitude* simply as N_c. CosMIn counts these modes which exit the Hubble radius during $a_P < a < a_X$ in the inflationary phase, re-enter during $a_X < a < a_Y$ in the intermediate phase and again exit during $a_Y < a < a_Q$ in the late-time accelerating phase. It is, in fact, possible to argue [5–7] that the cosmologically relevant part of the evolution is located inside the cosmic parallelogram $PXQY$.

Computing the cosmological constant using N_c now involves the following four steps:

(i) In the Friedmann equation for a spatially flat universe, $H^2(a) = H_\Lambda^2 + (C_R/a^4) + (C_m/a^3)$ where $C_R \equiv (8\pi G/3)a^4\rho_R(a), C_m \equiv (8\pi G/3)a^3\rho_m(a)$ are constants, we can eliminate C_R by rescaling a to $x \equiv a/C_R^{1/4}$ because N_c in Eq. (1) is invariant under this rescaling. This gives $H(a)$ as a function of two constants: H_Λ and $q \equiv C_m/C_R^{3/4} \propto \rho_m(a)/\rho_R(a)^{3/4} \propto (n_m/n_\gamma)$ which is expected to be determined by high energy physics.

(ii) We next determine x_2, x_1 corresponding to the epochs a_Y, a_X in Fig.1. The epoch $x_2 = x_2(H_\Lambda, q)$ is determined (as a function of H_Λ, q) by the condition that the tangent to the curve has unit slope, which is equivalent to $d[aH(a)]/da = 0$. It is straightforward to show from this, that x_2 satisfies

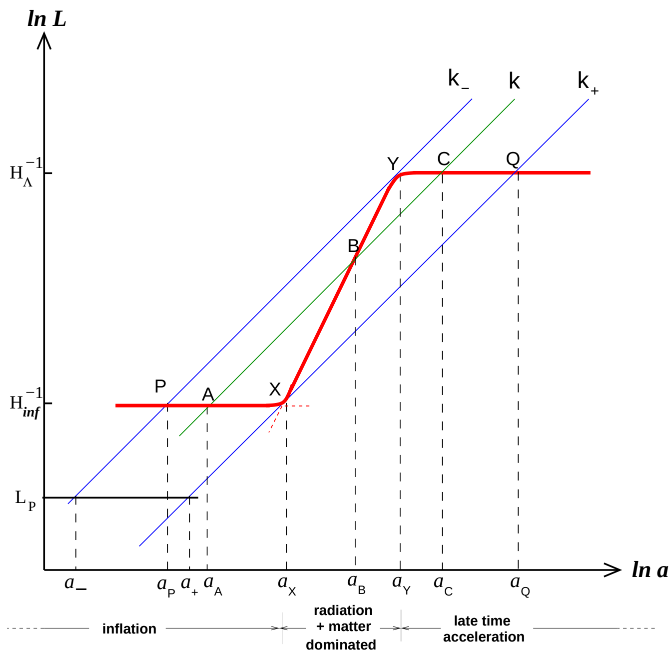

Figure 1: The Hubble radius $H^{-1}(a)$ of the universe (thick red line $PAXBYCQ$) delineates the three phases of evolution of our universe: (i) inflationary phase ($a < a_X, H^{-1} = H^{-1}_{inf}$ = constant); (ii) radiation/matter dominated phase ($a_X < a < a_Y, H^{-1} \propto a^2$ or $a^{3/2}$); (iii) late-time accelerated phase ($a_Y < a, H^{-1} = H^{-1}_\Lambda$ = constant). A generic mode with comoving wavenumber k and proper wavelength $\lambda_{\rm prop}(a) \equiv a/k \propto a$ (line ABC) exits the Hubble radius during inflation (A), re-enters during the radiation/matter dominated phase (B) and again exits during late-time acceleration (C). But modes with $k < k_-$ (line YP corresponds to $k = k_-$) or with $k > k_+$ (line XQ corresponds to $k = k_+$) cross the Hubble radius only once. The points P (obtained by drawing a unit slope tangent at Y and extending it backwards) and Q (obtained by drawing a unit slope line at X and extending it forwards) determine the cosmic parallelogram $PXQY$ of the relevant length scales [5–7]. The number of modes which cross the Hubble radius between X and Y (which is the same as the number of modes that cross during PX or YQ) is a unique number N_c for our universe, postulated to be 4π. These modes, when extrapolated back, cross the Planck length L_P during the interval $a_- < a < a_+$.

the quartic equation $2H_\Lambda^2 x_2^4 = 2 + qx_2$. To determine x_1, we need to connect the Hubble radius after inflationary reheating with its value during the inflationary phase, $H_{\rm inf}$. This will depend on the detailed modelling of the inflation, but, for our purpose, x_1 can be determined by matching $H(a)$ in the radiation dominated phase to the Hubble constant during the inflation, $H_{\rm inf} \equiv (\beta L_P)^{-1}$, giving $x_1 \simeq H_{\rm inf}^{-1/2} = (\beta L_P)^{1/2}$.

(iii) Once we know $x_1(\beta)$ and $x_2(H_\Lambda, q)$, we can express N_c in Eq. (1) as a function of H_Λ, q, β. Inverting this relation, we get the cosmological constant $\Lambda = 3H_\Lambda^2 = f(N_c, q, \beta)$ as a function of (N_c, q, β).

(iv) Given that the number of elements of Planck area contained on the surface of a Hubble sphere of radius L_P is $4\pi L_P^2/L_P^2 = 4\pi$, it can be argued that N_c should be of the order of 4π. (For more details on this aspect, related to the holographic evolution of the universe, see Refs. [4, 7].) This postulate determines the numerical value of Λ in terms of (q, β), both of which will be known from GUTs scale physics.

We will now describe the results obtained by the above procedure; the calculational details are given in the Appendix.

3 Numerical value of the cosmological constant

To test this idea, let us approximate the intermediate phase of the universe as purely radiation dominated (so that $q = 0$) and assume Planck scale inflation (so that $\beta = 1$) thereby eliminating *all* the free parameters. The above procedure now [4] gives $\Lambda = \Lambda(N_c, q = 0, \beta = 1)$ as a function of just N_c, as:

$$\Lambda L_P^2 = \frac{3}{4}\exp(-24\pi^2\mu); \qquad \mu \equiv \frac{N_c}{4\pi}\,. \tag{2}$$

Thus ΛL_P^2 is directly related to CosMIn and, in this simple model, there are *no other adjustable parameters*. Eq. (2) leads to the observed value $\Lambda L_P^2 = 3.4 \times 10^{-122}$ when $\mu = 1.18$. (Incidentally, the same analysis works whenever there is only a a single matter species; if $\rho \propto a^{-n}$, the exponential factor becomes $\exp[-12\pi^2\mu(n/(n-2))]$. The radiation dominated model $n = 4$ gives the value of μ closest to unity.)

The presence of matter ($q \neq 0$) and the fact that the inflationary scale may not be the Planck scale in our universe ($\beta \neq 1$), makes the algebra complicated but the procedure remains the same. Surprisingly, the postulate of $N_c = 4\pi$ works *better* in the real universe and reproduces the observed value of the cosmological constant when these complications are taken into account. We will now express the final result, first in a manner adapted to cosmology and then in the backdrop of high energy physics.

From a cosmological point of view, it is simpler to express Λ in terms of N_c, β and a *dimensionless* variable σ ($\propto q^{-1}$) rather than the *dimensionfull* quantity q. We will, therefore, motivate and define σ. Let $t = t_*, a = a_*$ denote an *arbitrary* epoch in the evolution of the universe (*not* necessarily today). Cosmologists living around $t = t_*$ would have defined the usual quantities $H(t_*)$,

critical density $\rho_c(t_*)$, and the density parameters $\Omega_m(t_*), \Omega_R(t_*)$ and written the Friedmann equation in terms of the three variables $[H(t_*), \Omega_m(t_*), \Omega_R(t_*)]$. (In the spatially flat universe, $\Omega_\Lambda(t_*) \equiv 1 - \Omega_m(t_*) - \Omega_R(t_*)$ is *not* an independent parameter.) But the numerical values of $[H(t_*), \Omega_m(t_*), \Omega_R(t_*)]$ depend on the epoch t_* while our expression for ΛL_P^2 can *only* depend on combinations of these parameters which are independent of t_*. It is, therefore, more transparent to work with such *epoch-invariant combinations* of these parameters.

One such set of three independent parameters is $[H_\Lambda, \sigma, \alpha]$ where $H_\Lambda = H(t_*)[1 - \Omega_m(t_*) - \Omega_R(t_*)]^{1/2}, \alpha \equiv [a_*\Omega_R(t_*)]/\Omega_m(t_*)$ and:

$$\sigma^4 \equiv \frac{\Omega_R^3(t_*)}{\Omega_m^4(t_*)}\,[1 - \Omega_m(t_*) - \Omega_R(t_*)] \tag{3}$$

This set $[H_\Lambda, \sigma, \alpha]$ contains the same information as the original set made of $[H(t_*), \Omega_m(t_*), \Omega_R(t_*)]$ but their numerical values are independent of the epoch $t = t_*$ at which they are defined. (Hence, of course, their numerical values can be determined by observations *today*.). We stress that, though σ^4 can also be expressed as $\sigma^4 = \rho_R^3\rho_\Lambda/\rho_m^4$ with an *apparent* dependence on $\rho_\Lambda \equiv 3H_\Lambda^2/8\pi G$ (which we are trying to determine from first principles!), it is actually *independent* of H_Λ, as can be clearly seen from Eq. (3). This fact is important to avoid any circuitous reasoning.

It is easy to show that, the solution to the Friedmann equation can be expressed in the form $a(t) = \alpha F(H_\Lambda t, \sigma)$ in terms of $(H_\Lambda, \alpha, \sigma)$. Because of the invariance of N with respect to the overall scaling of a, our expression for ΛL_P^2 will be independent of α. Following the steps (i) to (iv) described at the end of the previous section, we now get the expression $\Lambda(N_c, \sigma, \beta)$ to be:

$$\Lambda L_P^2 = \beta^{-2} C(\sigma) \exp[-24\pi^2\mu]; \qquad \mu \equiv \frac{N_c}{4\pi} \tag{4}$$

where $C(\sigma) = 12(\sigma r)^4\,(3r+4)^{-2}$ and r satisfies the quartic equation $\sigma^4 r^4 = (1/2)r + 1$. By taking (careful) limits, we can verify that Eq. (4) reduces to Eq. (2) in the special case of pure radiation and Planck scale inflation.

Given the numerical value of σ (which is determined through Eq. (3) at *any* epoch, say, today [1, 2]), the inflation scale determined by β, and our postulate $\mu = 1$, we can calculate the value of the cosmological constant from Eq. (4). For a GUTs scale inflation with $E_{\rm inf} = 1.2 \times 10^{15}$ GeV (corresponding to $\beta = 3.8 \times 10^7$) and $\sigma = 3.0 \times 10^{-3}$ (determined by cosmological observations), we get $\Lambda L_P^2 = 3.4 \times 10^{-122}$ which agrees with the observed value. The bulk of this "smallness" is contributed by the $\exp(-24\pi^2)$ factor in Eq. (4).

We now rephrase the above result in a manner adapted to high energy physics. By straightforward algebra, one can show that when $\sigma \ll 1$, Eq. (4) can be rewritten as:

$$L_P^4\rho_\Lambda = K\left(\frac{M_P}{m}\right)^2\left(\frac{n_\gamma}{n_m}\right)^2\beta^{-3}\exp(-36\pi^2\mu) \tag{5}$$

where $K \equiv (\pi^{11/2}/\zeta(3)^2)(1/6^3 10^{3/2}) \approx 0.055$, $\rho_m \equiv m n_m$ with m being the (mean) mass of the particle contributing to matter density and M_P is the Planck

mass. When viable particle physics models for (i) GUTs scale inflation, (ii) the dark matter candidate and (iii) baryogenesis are available, thereby determining β, m/M_P, and n_m/n_γ, Eq. (5) will predict the value of $L_P^4\rho_\Lambda$. In Eq. (5), the separation of effects due to the matter sector of the theory is evident. (Incidentally, the e-folding factor during the inflationary era in PX, when the cosmologically relevant modes exit the horizon, is just $6\pi^2 \approx 60$ in our model.)

The result in Eq. (4) is summarized in Fig.2. Since ΛL_P^2 in Eq. (4) depends only on the combination $\beta^{-2}\exp(-24\pi^2\mu)$, we get the same value of ΛL_P^2 (for a given σ) when this factor has a given value. Among all possible choices, $\mu = 1$ and $\beta = 1$, being natural, deserve special attention.

In the case of $\mu = 1$, for the range of σ allowed by cosmological observations, we get an acceptable range of values for ΛL_P^2 when β varies in the range $(2.6 - 7.3) \times 10^7$ (corresponding to $E_{\rm inf} = (0.84 - 1.4) \times 10^{15}$ GeV). This shows, gratifyingly, that for well accepted models of inflation, and for the acceptable range of cosmological parameters, our postulate $N_c = 4\pi$ gives the correct value for the cosmological constant.

The choice $\beta = 1$ corresponds to Planck scale inflationary models. When σ varies in the range allowed by observations, we find that μ varies between 1.144 and 1.153. Because ΛL_P^2 in Eq. (4) involves μ in the exponential, one might be skeptical about the narrowness of this range of variation. However, there are three conceptually attractive features about the Planck scale inflation modelled with $\beta = 1$.

First, it eliminates one free parameter, β, from the discussion and gives a direct relation between the two length scales Λ and L_P^2 which occur in the gravitational physics of the universe. Though the result has a dependence on σ, it is very weak and could be thought of as a matter of detail (like, for example, the fine structure correction to the spectral lines beyond Bohr's model). Second, in such a model, we think of the evolution in the intermediate phase (about which most cosmological investigations are concerned with!) as a mere transient connecting two de Sitter phases, both of which are semi-eternal. The fact that the de Sitter universe is time-translationally invariant makes it the natural candidate to describe the geometry of the universe dominated by a single length scale — L_P in the initial phase and $\Lambda^{-1/2}$ in the final phase. The quantum instability of the de Sitter phase at the Planck scale can lead to cosmogenesis and the transient radiation/matter dominated phase, which gives way, eventually, to the late-time acceleration phase. Finally, the argument for $N_c \approx 4\pi$ is quite natural with Planck scale inflation. It is obvious that the transition at X is entrenched in Planck scale physics in such a model, which can easily account for deviations of μ from unity. We believe this model deserves study in the context of candidate models of quantum gravity.

(As an aside, we mention the following: It is sometimes claimed in the literature that Planck scale inflation is ruled out because it produces too much of gravitational wave perturbations. What can be actually proved is that, *continuum* quantum field theory of spin-2 perturbations during inflation will lead to unacceptably large gravitational wave background, if the inflation scale is close to the Planck scale. But one cannot really [4] trust *continuum field theory* of the

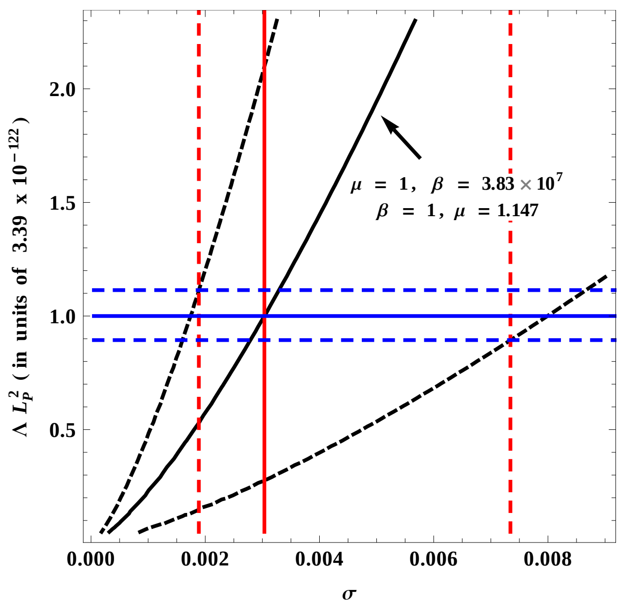

Figure 2: The postulate that $N_c = 4\pi\mu$ with μ being of order unity allows us to determine the numerical value of ΛL_P^2 in terms of (i) cosmological parameter $\sigma \equiv (\Omega_R^{3/4}/\Omega_m)\,(1-\Omega_m-\Omega_R)^{1/4}$ and (ii) the scale of inflation expressed through the dimensionless parameter $\beta = (H_{\rm inf} L_P)^{-1}$. Observational constraints [1, 2] lead to $\sigma = 0.003^{+0.004}_{-0.001}$, as shown by the three vertical (red) lines. The thick black curve is obtained from Eq. (4) if we take $\mu = 1$ (i.e., $N_c = 4\pi$) and $\beta = 3.826 \times 10^7$ (corresponding to the inflationary energy scale of $V_{\rm inf}^{1/4} = 1.164 \times 10^{15}$ GeV) and leads to the observed value of $\Lambda L_P^2 = 3.387 \times 10^{-122}$, shown by the horizontal unbroken (blue) line. The cosmologically allowed range in σ is bracketed by the two broken black curves obtained by varying β in the range $(2.641 - 7.286) \times 10^7$ (corresponding to the inflationary energy scale of $V_{\rm inf}^{1/4} = (0.84 - 1.4) \times 10^{15}$ GeV) with $N_c = 4\pi$ fixed. This gives the range $\Lambda L_P^2 = (3.028 - 3.773) \times 10^{-122}$, shown by the horizontal broken (blue) lines. Since our results only depend on the combination $\beta^{-2} \exp(-24\pi^2\mu)$, the same set of curves can also be incorporated in a Planck scale inflationary model ($\beta = 1$) with μ in the range $(1.144 - 1.153)$.

spin-2 field at Planck scales, based on which this result is obtained. There are suggestions [8] that this problem vanishes if corrections to propagators arising from a cut-off at the Planck scale are included.)

4 Discussion

The standard approach to cosmology, due to historical reasons, treats the evolution of the universe in a fragmentary manner. Planck scale physics, the inflationary era, the matter sector properties and the late-time acceleration each introduce parameters like $L_P^2,\ E_{\rm inf},\ (n_m/n_\gamma),\ \Lambda$ which are independently specified, bearing no relation with each other. Even if GUTs scale physics eventually determines $E_{\rm inf}$ and (n_m/n_γ), we still need a link between these parameters, L_P and Λ.

In the paradigm introduced here, the postulate $N_c = 4\pi$ acts as the connecting thread leading to a holistic approach to cosmic evolution, as is evident from Eq. (5). In a consistent quantum theory of gravity, we expect inflation (which determines β) and genesis of matter (which determines n_m/n_γ) to be related to Planck scale physics such that our fundamental relation in Eq. (5) (which is equivalent to $\sqrt{\Lambda}E_{\rm inf}^{-3}m(n_m/n_\gamma) = \text{constant}$) holds.

In this paper, we solve the cosmological constant problem *by actually determining* its numerical value in terms of other parameters. As far as we know, such an approach to this problem has not been attempted before. This approach may be thought of as being similar in spirit to the Bohr model of the hydrogen atom, which used the postulate $J = n\hbar$ to explain its energy levels. Here, our postulate $N_c = 4\pi$, captures the essence and explains the value of ΛL_P^2. We believe $N_c = 4\pi$ is simpler and more powerful than many other adhoc assumptions made in the literature [3] to solve the cosmological constant problem. *More importantly, we do know that this postulate is correct!* The value of CosMIn can be determined directly from the observed value of Λ *as well as* other cosmological parameters. We would then find that it is indeed very close to 4π.

The role of CosMIn is surprising because — in the standard inflationary paradigm — the crossing of the Hubble radius by modes has *no* physical significance and is merely a simple way to describe the behaviour of the perturbation equation in two limits. For example, instead of the crossing condition $aH = k$, one could have used $k/\pi, k/2\pi, ...$ on the right hand side. The importance of CosMIn is probably related to the the cosmic parallelogram $PXQY$ (see Fig.1) which arises *only* in a universe having three distinct phases. The epochs P and Q, limiting the otherwise semi-eternal de Sitter phases, now have a special significance [5–7]. Modes which exit the Hubble radius before $a = a_P$ never re-enter. On the other hand, the epoch $a = a_Q$ denotes (approximately) the time when the CMBR temperature falls below the de Sitter temperature [5–7]. The special role of $PXQY$ makes the value of CosMIn (which is the same for PX, XY or YQ) significant. As shown in Fig.1, these modes in $PXQY$ (with $k_- < k < k_+$) cross the Planck length during $a_- < a < a_+$, and it is likely that Planck scale

physics imposes the condition $N(a_-, a_+) \approx 4\pi$ at this stage in the correct quantum cosmological model. Clearly, this invites further work to examine the role of CosMIn and its numerical value from fundamental considerations.

Acknowledgements

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Appendix: Calculational details

This appendix contains some of the calculations leading to the results in Eq. (2), Eq. (4) and Eq. (5) in the main text. We will also comment on the *epoch invariant parameterization* of Friedmann cosmology which may be of interest by itself.

To study the evolution of a spatially flat universe containing a cosmological constant, pressure-free matter (which includes both dark matter and baryons) and radiation (which includes photons as well as any other relativistic species), one usually starts with the Friedmann equation written in the form:

$$\frac{\dot{a}^2}{a^2} = H_0^2\left[(1-\Omega_R-\Omega_m)+\frac{\Omega_R}{a^4}+\frac{\Omega_m}{a^3}\right] \tag{6}$$

Here, the Hubble parameter H_0 and the density parameters of matter and radiation, Ω_m and Ω_R are all evaluated at the present epoch ($t = t_0$). We have also chosen the spatial coordinates such that $a = 1$ at $t = t_0$. The cosmological constant, which we are after, is given in terms of these parameters as:

$$\Lambda \equiv 3H_\Lambda^2 = 3H_0^2(1-\Omega_R-\Omega_m) \tag{7}$$

This fact has two immediate consequences:

(a) If we allow the luxury of taking the values of *all* the three parameters $[H_0, \Omega_m, \Omega_R]$ from observations, we have nothing further to do to "determine" Λ! So, we need to treat at least one of these three as unknown. It makes sense to think of H_0 as unspecified, with $[\Omega_m, \Omega_R]$ given by observations. (We will say more about this a little later.)

(b) The numerical values of these parameters $[H_0, \Omega_m, \Omega_R]$ depend on the epoch $t = t_0$. Cosmologists living in a different era would have given different numerical values to these parameters. But, since Λ is a constant for the universe, the combination of the three parameters which occurs on the right hand side of Eq. (7) has the same numerical value at all epochs. This is an example of an *epoch invariant* combination of parameters for the universe; this concept will simplify our calculations both algebraically and conceptually in what follows.

Integrating Eq. (6) with the boundary condition $a = 0$ at $t = 0$ gives the expansion factor $a = a(H_0t, \Omega_m, \Omega_R)$ as a function of t, parameterized by the

set $[H_0, \Omega_m, \Omega_R]$. We can now determine the factor $(H_2 a_2 / H_1 a_1)$ which occurs in the right hand side of Eq. (1) as a function of $[H_0, \Omega_m, \Omega_R]$ and the inflation scale β. The postulate $N_c = 4\pi\mu$ with $\mu \approx 1$ will give an equation connecting the four parameters $[H_0, \Omega_m, \Omega_R, \beta]$. Inverting this relation, we can find, say, H_0 in terms of the other parameters and thus the cosmological constant in terms of $[\Omega_m, \Omega_R, \beta]$. Substituting this into Eq. (7), we can determine Λ in terms of the other parameters.

While it is straightforward to perform this exercise (we have indeed done it as a cross check on the calculation) there is a nicer and conceptually better way to proceed. To appreciate this, note that there is a constraint on how the final expression we obtain for Λ can depend on the parameters Ω_m and Ω_R. As we mentioned before, cosmologists determining these parameters at a different epoch would have assigned different numerical values to these parameters, but the value of Λ cannot depend on the epoch at which Ω_m and Ω_R are measured! Further, we know that the right hand side of Eq. (1) is independent of rescaling a by a factor. This freedom can be used to remove one parameter from the set. Therefore, the final expression for Λ can only depend on the inflation scale β and a *particular* combination of Ω_m and Ω_R which is independent of the epoch at which they are measured. Clearly, it is more efficient for our purpose to re-parameterize the Friedmann equation in terms of such epoch invariant parameters, right from the start.

This can be done in many ways and we will choose the following procedure. Let us consider cosmologists working at the epoch $t = t_*$ (corresponding to $a(t_*) \equiv a_*$). They would have defined the usual quantities $H(t_*)$, the critical density $\rho_c(t_*)$, and the density parameters $\Omega_m(t_*), \Omega_R(t_*)$ and written the Friedmann equation in terms of the three variables $[H(t_*), \Omega_m(t_*), \Omega_R(t_*)]$ as:

$$\frac{\dot{a}^2}{a^2} = H(t_*)^2 \left[[1 - \Omega_m(t_*) - \Omega_R(t_*)] + \frac{a_*^3 \Omega_m(t_*)}{a^3} + \frac{a_*^4 \Omega_R(t_*)}{a^4} \right] \tag{8}$$

Using the definition of H_Λ in Eq. (7), we can write this equation as

$$\left(\frac{\dot{a}}{a}\right)^2 = H_\Lambda^2 + \frac{C_m(t_*)}{a^3} + \frac{C_R(t_*)}{a^4} \tag{9}$$

where

$$C_m(t_*) = \frac{a_*^3 \Omega_m(t_*) H_\Lambda^2}{1 - \Omega_m(t_*) - \Omega_R(t_*)}\, ; \quad C_R(t_*) = \frac{a_*^4 \Omega_R(t_*) H_\Lambda^2}{1 - \Omega_m(t_*) - \Omega_R(t_*)} \tag{10}$$

We have now introduced the set of three parameters $[H_\Lambda, C_m(t_*), C_R(t_*)]$ defined at $t = t_*$ instead of the original set $[H_0, \Omega_m, \Omega_R]$ defined at the current epoch. While H_Λ is clearly independent of the epoch t_*, the other two parameters are not. This is easily taken care of by introducing the variable $x(t) \equiv a(t)/\alpha$ where α is defined by

$$\alpha = \frac{a_* \Omega_R(t_*)}{\Omega_m(t_*)} = \frac{a \rho_R(a)}{\rho_m(a)} \tag{11}$$

Equation (9) now becomes an equation for $x(t)$:

$$\left(\frac{\dot{x}}{x}\right)^2 = H_\Lambda^2 \left[1 + \frac{1}{\sigma^4}\left(\frac{1}{x^3} + \frac{1}{x^4}\right)\right] \tag{12}$$

where σ is defined through:

$$\sigma^4 \equiv \frac{C_R^3 H_\Lambda^2}{C_m^4} = \frac{\rho_R^3}{\rho_m^4}\,\rho_\Lambda = \frac{\Omega_R^3(t_*)}{\Omega_m^4(t_*)}\left[1 - \Omega_m(t_*) - \Omega_R(t_*)\right] \tag{13}$$

with $\rho_\Lambda \equiv 3H_\Lambda^2/8\pi G$.

We have now re-written the Friedmann equation in terms of the three parameters $[H_\Lambda, \alpha, \sigma]$ all of which are independent of the epoch t_* at which they are defined. The parameter α is obviously independent of the epoch t_*, since both $a\rho_R(a)$ and $\rho_m(a)$ scale as a^{-3}, making the ratio independent of a. (This is, however, only of academic interest; our result cannot depend on the parameter α because of the rescaling invariance on a mentioned above.) The epoch independence of σ is obvious from the last equality in Eq. (13), because both ρ_Λ and ρ_R^3/ρ_m^4 are independent of t_*. Another significant fact is that σ is independent of H_Λ, and depends only on the parameters $\Omega_m(t_*)$ and $\Omega_R(t_*)$, in spite of the first two equalities in Eq. (13) which have H_Λ and ρ_Λ in them. This is clear from the last expression in Eq. (13).

Using this parameterization, we can now express the right hand side of Eq. (1) in terms of $[H_\Lambda, \sigma, \beta]$. Our postulate $N_c \approx 4\pi$ now gives a relationship among these parameters, inverting which we can express H_Λ (and thus the cosmological constant) in terms of β and σ, both of which, of course, can be specified independently of H_Λ. This is the strategy we will adopt.

We begin by determining x_2 and x_1. It is easy to see that the condition $d[aH(a)]/da = 0$, which determines x_2, allows us to express x_2 as a function of σ, say, as $x_2 = f(\sigma)$, where the function $f(\sigma)$ satisfies the quartic equation:

$$2\sigma^4 f^4 = 2 + f; \qquad x_2 = f(\sigma)\,. \tag{14}$$

On the other hand, x_1 is determined by matching $H(a)$ with the Hubble constant during the inflation $H_{\rm inf}$. Since this occurs in the radiation dominated phase, we only need to retain the x^{-4} term in the right hand side of Eq. (12). This gives

$$H_{\rm inf}^2 \cong \frac{H_\Lambda^2}{\sigma^4 x_1^4}; \qquad x_1^4 = \frac{H_\Lambda^2}{\sigma^4 H_{\rm inf}^2} \tag{15}$$

We next compute $H_1^2 x_1^2$ and $H_2^2 x_2^2$. We have, from Eq. (15):

$$H_1^2 x_1^2 = \frac{H_{\rm inf} H_\Lambda}{\sigma^2} \tag{16}$$

Using Eq. (14), we can express $H_2^2 x_2^2$ as:

$$H_2^2 x_2^2 = H_\Lambda^2 \left[f^2 + \frac{1}{\sigma^4}\left(\frac{1}{f} + \frac{1}{f^2}\right)\right] = \frac{H_\Lambda^2}{2\sigma^4 f^2}\,[4 + 3f]\,, \tag{17}$$

giving the ratio:

$$\frac{H_2^2 x_2^2}{H_1^2 x_1^2} = \left(\frac{H_\Lambda}{H_{\rm inf}}\right) \frac{(4+3f)}{2\sigma^2 f^2}\,. \tag{18}$$

Substituting this in Eq. (1) and writing $N_c = 4\pi\mu$ we get our final result

$$e^{-12\pi^2\mu} = \beta(H_\Lambda\, L_P)\left(\frac{4+3f}{2\sigma^2 f^2}\right) \tag{19}$$

This is equivalent to the result quoted in the text:

$$\Lambda L_P^2 = \beta^{-2} C(\sigma) \exp[-24\pi^2\mu]; \qquad C(\sigma) = 12(\sigma f)^4\,(3f+4)^{-2} \tag{20}$$

where f satisfies the quartic equation $\sigma^4 f^4 = (1/2)f + 1$.

As we stressed before, σ is an epoch independent parameter defined through Eq. (13) and its value is, of course, independent of H_Λ, which we are trying to determine. Since σ is epoch independent, we can determine its possible value and the range from current observations. The quartic equation requires numerical solution and the result is plotted in Fig. 2.

The quartic equation, Eq. (14) can be solved approximately in two limits, viz., $\sigma \gg 1$ and $\sigma \ll 1$. In the first case ($\sigma \gg 1$) the approximate solution to the quartic in Eq. (14) is given by $f \approx \sigma^{-1}$. In this limit, to the same order of accuracy, Eq. (19) becomes $e^{-12\pi^2\mu} = 2\beta H_\Lambda L_P$ which can be rewritten as

$$\Lambda L_P^2 = \frac{3}{4}\,\beta^{-2}\,\exp(-24\pi^2\mu) \tag{21}$$

It is easy to verify that this limit corresponds to a purely radiation dominated universe with $\Omega_m(t_*) \to 0$. As mentioned earlier, our expression for Λ can *only* depend on $\Omega_m(t_*)$ and $\Omega_R(t_*)$ through σ; when $\sigma \gg 1$ this dependence drops out.

This result can also be understood from the fact that, when $\Omega_m(t_*) = 0$, the Friedmann equation, Eq. (9), becomes $H^2(a) = H_\Lambda^2 + C_R(t_*)a^{-4}$. The re-scaling freedom in a will allow us to eliminate $C_R(t_*)$ and hence the result can only depend on β. (This is, in fact, a general feature. It can be shown that, in a universe containing a single matter species in addition to a cosmological constant, all the dependence on cosmological parameters pertaining to the matter species can be scaled out. Further, if the energy density of the matter species varies as a^{-n}, the exponential factor in Eq. (21) becomes $\exp[-12\pi^2\mu n(n-2)^{-1}]$.)

In the other limit of $\sigma \ll 1$, the quartic equation, Eq. (14), has the approximate solution $f \approx (2\sigma^4)^{-1/3}$, and Eq. (19) gives:

$$e^{-12\pi^2\mu} = 3\beta L_P H_\Lambda \left(\frac{1}{2\sigma}\right)^{2/3} \tag{22}$$

In this case, we can also express the result in terms of the ratio between the number densities of photons and matter particles. To do this, we raise both sides of the above equation to the third power, obtaining

$$e^{-36\pi^2\mu} = \frac{9}{4}\beta^3(\Lambda L_P^2)\, L_P \left(\frac{H_\Lambda}{\sigma^2}\right) = \frac{9}{4}\beta^3\Lambda\left(\frac{H_\Lambda}{\sigma^2}\right) \tag{23}$$

In arriving at the last equality we have used natural units with $G = 1, \hbar = 1$ and $c = 1$ so that $L_P = 1$. Using the second equality in Eq. (13), we find that

$$\frac{H_\Lambda^2}{\sigma^4} = \frac{\rho_m^4}{\rho_R^3\rho_\Lambda}\frac{8\pi}{3}\rho_\Lambda = \frac{8\pi}{3}\frac{\rho_m^4}{\rho_R^3} \tag{24}$$

which, when substituted into Eq. (23), gives the result:

$$\Lambda = \frac{4}{9}\beta^{-3}\left(\frac{8\pi}{3}\right)^{-1/2}\left(\frac{\rho_R^{3/4}}{\rho_m}\right)^2 e^{-36\pi^2\mu} \tag{25}$$

The combination $\rho_R^{3/4}/\rho_m$ is essentially the ratio between the number densities of photons and matter particles, and can be written in the form

$$\frac{\rho_R^{3/4}}{\rho_m} = \left(\frac{\pi^2}{15}T^4\right)^{3/4}\frac{1}{mn_m} = \left(\frac{\pi^2}{15}\right)^{3/4}\left(\frac{\pi^2}{2\zeta(3)}\right)\left(\frac{n_\gamma}{mn_m}\right) \tag{26}$$

where we have used the relation $n_\gamma = [2\zeta(3)/\pi^2]T^3$ and $\rho_m \equiv mn_m$ with m denoting the mean mass of the matter particles contributing to the pressure-free matter density. (This will be dominated by dark matter with baryons contributing a small fraction.) Substituting all these into Eq. (25), and writing $\Lambda \equiv 8\pi\rho_\Lambda$, we get Eq. (5) of the main text:

$$L_P^4\rho_\Lambda = K\left(\frac{M_P}{m}\right)^2\left(\frac{n_\gamma}{n_m}\right)^2\beta^{-3}\exp(-36\pi^2\mu) \tag{27}$$

where $K \equiv (\pi^{11/2}/\zeta(3)^2)(1/6^3 10^{3/2}) \approx 0.055$. In the last expression, we have re-introduced the Planck mass M_P appropriately. This expression shows a clear separation of the effects due to high energy physics and that due to our postulate that CosMIn has the numerical value 4π. The exponential factor $\exp(-36\pi^2) \approx 10^{-154.3}$ is now too tiny compared to the observed value of $\rho_\Lambda L_P^4 = \Lambda L_P^2/8\pi \approx 1.34 \times 10^{-123}$; we are saved by two large factors multiplying the exponential: $(M_P/m)^2 \approx 1.48 \times 10^{34}$ if we take the dark matter particle mass to be $m \approx 100$ GeV, and $(n_\gamma/n_m)^2 \approx 1.87 \times 10^{21}$. These two factors, along with β^{-3}, lead to the correct result.

High energy physics models will eventually determine the inflationary scale β, the mass m of the dark matter candidate particle, its abundance relative to photons, and the baryon-to-photon ratio. Given these numbers, everything on the right hand side of Eq. (27) can be computed, thereby determining the cosmological constant.

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